Exercise 7.2.12

The motion of a body falling in a resisting medium may be described by

$$m\frac{dv}{dt} = mg - bv$$

when the retarding force is proportional to the velocity, v. Find the velocity. Evaluate the constant of integration by demanding that $v(0) = 0$.

Solution

Method I: Separation of Variables

Solve the ODE by separating variables. Divide both sides by $m(mg - bv)$ and bring dt to the right side.

$$\frac{dv}{mg - bv} = \frac{dt}{m}$$

Integrate both sides.

$$\int^v \frac{dr}{mg - br} = \int^t \frac{ds}{m} + C_1$$

Evaluate the integral on the right and make the substitution $u = mg - br$ $(du = -b\,dr)$ on the left.

$$\int^{mg-bv} \frac{1}{u}\left(\frac{du}{-b}\right) = \frac{t}{m} + C_1$$

$$-\frac{1}{b}\int^{mg-bv} \frac{du}{u} = \frac{t}{m} + C_1$$

$$-\frac{1}{b}\ln|u|\Big|^{mg-bv} = \frac{t}{m} + C_1$$

$$-\frac{1}{b}\ln|mg - bv| = \frac{t}{m} + C_1$$

Multiply both sides by $-b$, using a new constant C_2 for $-bC_1$.

$$\ln|mg - bv| = -\frac{bt}{m} + C_2$$

Exponentiate both sides.

$$\begin{aligned} |mg - bv| &= e^{-bt/m + C_2} \\ &= e^{-bt/m}e^{C_2} \end{aligned}$$

Remove the absolute value sign on the left by placing $\pm$ on the right.

$$mg - bv = \pm e^{C_2}e^{-bt/m}$$

Use a new constant C_3 for $\pm e^{C_2}$.

$$mg - bv = C_3 e^{-bt/m} \tag{1}$$

Now apply the initial condition $v(0) = 0$ to determine C_3.

$$mg - b(0) = C_3 e^{-b(0)/m}$$

$$mg = C_3$$

Substitute this formula for C_3 into equation (1).

$$mg - bv = mge^{-bt/m}$$

Solve for v.

$$\begin{aligned} bv &= mg - mge^{-bt/m} \\ &= mg(1 - e^{-bt/m}) \end{aligned}$$

Therefore,

$$v(t) = \frac{mg}{b}(1 - e^{-bt/m}).$$

Method II: Integrating Factor

Here the ODE will be solved by multiplying both sides by an integrating factor.

$$m\frac{dv}{dt} = mg - bv$$

Bring bv to the left side.

$$m\frac{dv}{dt} + bv = mg$$

Divide both sides by m so that the coefficient of dv/dt is 1.

$$\frac{dv}{dt} + \frac{b}{m}v = g$$

The integrating factor to be used is

$$I = \exp\left(\int^t \frac{b}{m}\,ds\right) = e^{bt/m}.$$

Multiply both sides of the previous equation by I.

$$e^{bt/m}\frac{dv}{dt} + \frac{b}{m}e^{bt/m}v = ge^{bt/m}$$

The left side can now be written as a derivative by the product rule.

$$\frac{d}{dt}(e^{bt/m}v) = ge^{bt/m}$$

Integrate both sides with respect to t.

$$e^{bt/m}v = \frac{mg}{b}e^{bt/m} + C_4$$

Divide both sides by $e^{bt/m}$.

$$v(t) = \frac{mg}{b} + C_4 e^{-bt/m}$$

Apply the initial condition $v(0) = 0$ to determine C_4.

$$0 = \frac{mg}{b} + C_4 e^{-b(0)/m} \quad \rightarrow \quad C_4 = -\frac{mg}{b}$$

Therefore,

$$\begin{aligned} v(t) &= \frac{mg}{b} - \frac{mg}{b} e^{-bt/m} \\ &= \frac{mg}{b}(1 - e^{-bt/m}). \end{aligned}$$